GLOBAL BEHAVIOR OF THE DIFFERENCE EQUATION $x_{n+1} = a + \sum_{i=0}^{k} b_i \frac{x_{n-(2i+1)}}{x_{n-2i}}$

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Abstract— The main objective of this paper is to study the qualitative behavior for a class of nonlinear rational difference equation. We study the local stability, periodicity, Oscillation, boundedness, and the global stability for the positive solutions of equation. Examples illustrate the importance of the results

Keywords— Difference equation, stability, oscillation, boundedness, globale stability and Periodicity

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1 Introduction

In this paper we deal with some properties of the solutions of the difference equation

$$x_{n+1} = a + \sum_{i=0}^{k} b_i \frac{x_{n-(2i+1)}}{x_{n-2i}}, \; n = 0,1,2,..., \qquad (1.1)$$

Where the initial conditions $x_{-j}, x_{-j+1},..., x_0$ are arbitrary positive real numbers where $j = (2k+1)$. There is a class of nonlinear difference equations, known as the rational difference equations, each of which consists of the ratio of two polynomials in the sequence terms in the same from. There has been a lot of work concerning the global asymptotic of solutions of rational difference equations [1], [3], [4]. [5], [7], [8] and [10].

Let I be an interval of real numbers and let

$$F : I^{j+1} \to I,$$

where F is a continuous function. Consider the difference equation

$$x_{n+1} = F(x_n, x_{n-1},..., x_{n-j}), \; n = 0,1,2,..., \qquad (1.2)$$

With the initial condition $x_{-j}, x_{-j+1},..., x_0 \in I.$

Definition 1. *[2] (Equilibrium Point)*

A point $\bar{x} \in I$ is called an equilibrium point of Eq. (2r) if

$$\bar{x} = f(\bar{x}, \bar{x},..., \bar{x}).$$

That is, $x_n = \bar{x}$ for $n \geq 0,$ is a solution of Eq. (2r), or equivalently, $\bar{x}$ is a fixed point of

Definition 2. *[6] (Stability)*

Let $\bar{x} \in (0,\infty)$ be an equilibrium point of Eq. (1.2) Then

i. An equilibrium point $\bar{x}$ of Eq. (1.2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-j}, x_{-j+1},..., x_0 \in (0,\infty)$ with $|x_{-j} - \bar{x}| + |x_{-j+1} - \bar{x}| + ... + |x_0 - \bar{x}| < \delta,$ then $|x_n - \bar{x}| < \varepsilon$ for all $n \geq -j$.

ii. An equilibrium point of Eq. (1.2) is called locally asymptotically stable if $\bar{x}$ is locally stable and there exists such that, if with then

$$\lim_{n\to\infty} x_n = \bar{x}.$$

iii. An equilibrium point $\bar{x}$ of Eq. (1.2) is called a global attractor if for every $x_{-j}, x_{-j+1},..., x_0 \in (0,\infty)$ we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

iv. An equilibrium point $\bar{x}$ of Eq. (1.2) is called globally asymptotically stable if $\bar{x}$ is locally stable and a global attractor.

v. An equilibrium point $\bar{x}$ of Eq. (1.2) is called unstable if $\bar{x}$ is not locally stable.

Definition 3. *[11] (Permanence)*

Eq. (1.2) is called permanent if there exists numbers m and M with $0 < m < M < \infty$ such that for any initial conditions $x_{-j}, x_{-j+1},..., x_0 \in (0,\infty)$ there exists a positive integer N which depends on the initial conditions such that

$$m \leq x_n \leq M \text{ for all } n \geq N.$$

Definition 4. *[9] (Periodicity)*

A sequence $\{x_n\}_{n=-j}^{\infty}$ is said to be periodic with period if $x_{n+p} = x_n$ for all $n \geq -r.$ A sequence $\{x_n\}_{n=-j}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

The linearized equation of Eq. (1.2) about the equilibrium point $\bar{x}$ is defined by the equation

$$z_{n+1} = \sum_{i=0}^{j} p_i z_{n-(2i+1)} = 0, \qquad (1.3)$$

where

$$p_i = \frac{\partial F(\bar{x}, \bar{x},..., \bar{x})}{\partial x_{n-j}}, \; i = 0,1,..., j.$$

The characteristic equation associated with Eq. (1.3) is

$$\lambda^{j+1} - p_0\lambda^j - p_1\lambda^{j-1} - ... - p_{j-1}\lambda - p_j = 0. \quad (1.4)$$

Theorem 1.1 *[2]. Assume that   is a* C^1 *function and let* $\bar{x}$ *be an equilibrium point of Eq. (1.2). Then the following statements are true:*

i. *If all roots of Eq. (1.4) lie in the open unit disk* $|\lambda| < 1$ *, then he equilibrium point* $\bar{x}$ *is locally asymptotically stable.*

ii. *If at least one root of Eq. (1.4) has absolute value greater than one, then the equilibrium point* $\bar{x}$ *is unstable.*

iii. *If all roots of Eq. (1.4) have absolute value greater than one, then the equilibrium point* $\bar{x}$ *is a source.*

Theorem 1.2 *[6] Assume that* $p_i \in R, i = o,1,2,..., j.$ *Then*

$$\sum_{i=1}^{j} |p_i| < 1,$$

is a sufficient condition for the asymptoticcally stable of Eq. (1.5)

$$p_0 y_{n+j} + p_1 y_{n+j-1} + ... + p_j y_n = 0, \; n = 0,1,.... \quad (1.5)$$

2 Local stability of the equilibrium point

In this section we investigate the local stability character of the solutions of Eq. (1.1). has a unique nonzero equilibrium point

$$\bar{x} = a + \sum_{i=0}^{k} b_i \frac{\bar{x}}{\bar{x}},$$

$$\bar{x} = a + B.$$

Then

$$B = \sum_{i=0}^{k} b_i.$$

Let $f : (0,\infty)^{2k+1} \to (0,\infty)$ be a function defined by

$$f(u_0, u_1, ..., u_{2k+1}) = a + \sum_{i=0}^{k} b_i \frac{u_{2i+1}}{u_{2i}}.$$

Therefore it follows that

$$\frac{\partial f(u_0, u_1, ..., u_{2m+1})}{\partial u_{2m}} = -b_m \frac{u_{2m+1}}{(u_{2m})^2},$$

and

$$\frac{\partial f(u_0, u_1, ..., u_{2m+1})}{\partial u_{2m+1}} = b_m \frac{1}{u_{2m}}.$$

Then we see that

$$\frac{\partial f(\bar{x}, ..., \bar{x})}{\partial u_{2m}} = \frac{-b_m}{a+B} = P_{2m},$$

and

$$\frac{\partial f(\bar{x}, ..., \bar{x})}{\partial u_{2m+1}} = \frac{b_m}{a+B} = P_{2m+1}.$$

Then the linearized equation of Eq. (1.1) about $\bar{x}$ is

$$z_{n+1} = \sum_{i=0}^{2k+1} P_i z_{n-i}.$$

Theorem 2.1 *Assume that*

$$B < a.$$

Than the equilibrium point of Eq. (1.1) is locally stable.

Proof It is follows by Theorem 1.2 that, Eq. (2.2) is locally stable if

$$|p_0| + |p_1| + ... + |p_{2k+1}| < 1.$$

That is

$$\left|\frac{-b_0}{a+B}\right| + \left|\frac{b_0}{a+B}\right| + \left|\frac{-b_1}{a+B}\right| + \left|\frac{b_1}{a+B}\right| + ... + \left|\frac{-b_k}{a+B}\right| < 1.$$

$$\frac{2B}{a+B} < 1,$$

then

$$2B < a + B,$$

thus

$$B < a.$$

Hence, the proof is completed.

Example 2.1 *Fig. 1, shows that Eq. (1.1) has Local stable solutions if* $a = 4, \; b_0 = 1, \; b_1 = 0.5, \; b_2 = 0.3, \; \bar{x} = 5.8.$

3 PERIODIC SOLUTIONS

In this part of the research we are studying the possibility of the existence of periodic solutions to the Eq. (1.1).

Theorem 3.1 *The Eq. (1.1) has no prime period-two solutions.*
Proof Suppose that there exists a prime period-two solution

$$\ldots, p, q, p, q, p, q, \ldots$$

If $x_n = x_{n-2} = \ldots = x_{n-2k} = q,$

$x_{n+1} = x_{n-1} = \ldots = x_{n-(2k+1)} = p$

$$pq = \alpha q + Bp, \qquad (3.1)$$

also,

$$pq = \alpha p + Bq. \qquad (3.2)$$

By (3.1) and (3.2), we have

$$(p-q)(a+B) = 0 \quad \Rightarrow \quad p = q$$

Hence, the proof is completed.

4 OSCILLATORY SOLUTION

Theorem 4.1 *Eq. (1.1) has an oscillatory solution.*
Proof $(First)$ assume that,

$$x_{-(2k+1)}, x_{-(2k+3)}, \ldots, x_{-1} > \bar{x} \quad and \quad x_{-2k}, x_{-2k+2}, \ldots, x_0 < \bar{x},$$

so

$$x_1 = a + \sum_{i=0}^{k} b_i \frac{x_{-(2k+1)}}{x_{-2k}},$$

then

$$x_1 > \alpha + \sum_{i=0}^{k} b_i \frac{\bar{x}}{\bar{\bar{x}}},$$

and

$$x_1 > \alpha + B = \bar{x}.$$

So, we have

$$x_2 = a + \sum_{i=0}^{k} b_i \frac{x_{-2k}}{x_{-(2k+1)}},$$

so,

$$x_2 < \alpha + \sum_{i=0}^{k} b_i \frac{\bar{x}}{\bar{\bar{x}}},$$

then,

$$x_2 < \alpha + B = \bar{x}.$$

$(Secandiy)$ assume that,

$$x_{-(2k+1)}, x_{-(2k+3)}, \ldots, x_{-1} < \bar{x} \quad and \quad x_{-2k}, x_{-2k+2}, \ldots, x_0 > \bar{x},$$

then, so

$$x_1 = a + \sum_{i=0}^{k} b_i \frac{x_{-(2k+1)}}{x_{-2k}},$$

then

$$x_1 < \alpha + \sum_{i=0}^{k} b_i \frac{\bar{x}}{\bar{\bar{x}}},$$

and

$$x_1 < \alpha + B = \bar{x}.$$

So, we have

$$x_2 = a + \sum_{i=0}^{k} b_i \frac{x_{-2k}}{x_{-(2k+1)}},$$

so,

$$x_2 > \alpha + \sum_{i=0}^{k} b_i \frac{\bar{x}}{\bar{\bar{x}}},$$

then,

$$x_2 > \alpha + B = \bar{x}.$$

One cam proceed in prove manger to show that $x_3 < \bar{x}$ and $x_4 > \bar{x}$ and soon. Hence, the proof is completed.

Example 4.1 *Fig. 2, shows that Eq. (1.1) has 0scilatory solution if*
$a = 2, \quad b_0 = 1, \quad b_1 = 0.05, \quad b_2 = 1.1, \quad \bar{x} = 4.15.$.
$(\text{see Table } 1)$

5 Boundedness

Our aim in this section we investigate the boundedness of the positive solutions of Eq. (1.1).

Theorem 5.1 *Let* $\{x_n\}$ *be a solution of Eq. (1.1). Then the following statements are true:*

1) *Suppose* $B<1$ *and for some* $N \geq 0,$ *the initial condition*

$$x_{N-k+1},..., x_{N-1}, x_N \in [B,1],$$

then we have

$$a+B^2 \leq x_n \leq a+1, \text{ for all } n \geq N.$$

2) *Suppose* $B>1$ *and for some* $N \geq 0,$ *the initial condition*

$$x_{N-k+1},..., x_{N-1}, x_N \in [1,B],$$

then

$$a+1 \leq x_n \leq a+B^2, \text{ for all } n \geq N.$$

Proof $(First)$ For some $N \geq 0,$ $B<1$ and we have

$$x_{n+1} = a + \sum_{i=0}^{k} b_i \frac{x_{n-(2k+1)}}{x_{n-2k}},$$

we have,

$$\leq a + \sum_{i=0}^{k} b_i \frac{1}{B}.$$

$$\leq a+1.$$

Then we have

$$x_{n+1} = a + \sum_{i=0}^{k} b_i B,$$

$$\geq a + B^2.$$

so,

$$a+B^2 \leq x_n \leq a+1.$$

Similarly, we can prove other cases which is omitted here for convenience. Hence, the proof is completed.

6 Global stability

Our aim in this section we investigate the global stability of Eq. (1.1).

Theorem 6.1 *If* $a \neq 1$ *and* $G < 2c_k$ *then the equilibrium point* $\bar{x}$ *of Eq.(1r) is global attractor.*

Proof Let is increasing in u_{2m+1} and decreasing in u_{2m}.

Suppose that (m, M) is a solution of the system

$$m = f(M, m, M, m,..., m) \text{ and } M = f(m, M, m, M,..., M).$$

Then

$$m = a + \sum_{i=0}^{k} bi \frac{m}{M},$$

$$M = a + \sum_{i=0}^{k} bi \frac{M}{m}.$$

We conclude from the above,

$$mM = aM + Bm,$$

and

$$mM = am + BM.$$

Subtract (6.1) from (6.2) deduce,

$$(a+B)(m-M) = 0.$$

Since $a+B \neq 0$ then

$$M = m$$

Hence, the proof is completed.

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